

Quasi-degenerate neutrinos and tri-bi-maximal mixing

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Abstract

We consider how, for quasi-degenerate neutrinos with tri-bi-maximal mixing at a high-energy scale, the mixing angles are affected by radiative running from high to low-energy scales in a supersymmetric theory. The limits on the high-energy scale that follow from consistency with the observed mixing are determined. We construct a model in which a non-Abelian discrete family symmetry leads both to a quasi-degenerate neutrino mass spectrum and to near tri-bi-maximal mixing.

1 Introduction

Neutrino oscillation data is at present consistent [1, 2] with just three light neutrinos with near tri-bi-maximal (TBM) mixing between flavours [3, 4, 5, 6, 7]. However the nature of the mass spectrum is still not established, being consistent with either a normal or an inverted hierarchy. Moreover, although the magnitude of the mass squared difference between neutrinos is reasonably well determined, the absolute scale of mass is not, being consistent with both a strongly hierarchical spectrum or a quasi-degenerate (QD) spectrum.

Radiative running is especially important for QD neutrinos, as the effects on mixing angles are larger for QD neutrinos than in the hierarchical case. This was stressed in [8, 9] where the mixing favoured at the time, bi-maximal mixing, was studied in-depth. More recent studies of mixing angles running include [10, 11, 12, 13] (and references therein). Here we discuss radiative corrections to TBM mixing, assuming that it arises through new physics, such as a family symmetry, at a high-energy scale. We determine how high, in a supersymmetric extension of the Standard Model, the initial energy scale can be while maintaining near TBM mixing at the low-energy scales relevant to oscillation experiments. The main difference from existing work is that emphasis is placed on the energy scales rather than on the resulting low-energy angles. Specifically, we set the angles to their TBM values at high-energy scales, run the angles to low-energy and iterate the process to find the highest-energy scale that still keeps the low-energy angles within current experimental bounds. The process is then repeated for different points of the parameter space, and the results are presented as a contour plot in the $m_{\nu_i} - \tan \beta$ plane ($i = 1$ for normal and $i = 3$ for inverted hierarchy).

The underlying question raised by the observed near TBM mixing is the origin of the pattern and the reason it is so different from quark mixing. Models based on family symmetries, particularly discrete non-Abelian family symmetries, have been constructed to explain this pattern, e.g. [14, 15]. In these models the difference between the quark and lepton sector follows naturally from the see-saw mechanism together with a strongly hierarchical right-handed neutrino Majorana mass spectrum. However these models only apply to the case of an hierarchical neutrino mass spectrum. Here we discuss how a discrete non-Abelian family symmetry can also give rise to near TBM mixing for the case of a QD spectrum.

2 Radiative corrections to TBM mixing

Family-symmetry models are typically constructed at some high scale, M_F , at which the model specifies relationships among parameters. To compare the predictions to low-energy data, radiative effects should be

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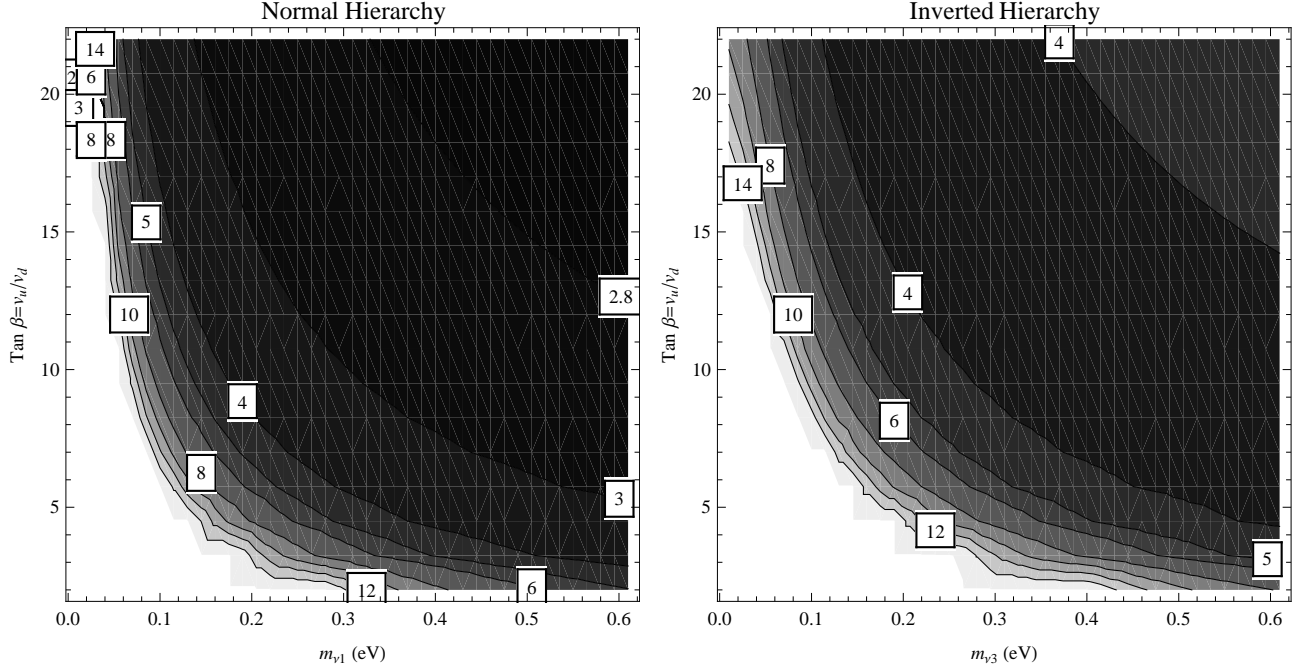


Figure 1: Shows contours of $\text{Log}_{10}(M_F)$ where M_F is the highest-energy family-symmetry breaking scale at which we can set TBM and have the neutrino mixing within 4σ of the low-energy observed values. The white region in the lower left of the contour plots are the regions where M_F can be greater than 10^{16} GeV.

considered through the use of the renormalization group equations. When there is a strong hierarchy, it is often the case that these running effects do not change the mixing angles by much [10, 11, 12, 13]. In the case of QD neutrinos, however, the mixing angles can change a lot with the energy scale, to the point of erasing any special structure arranged by a family symmetry. For model-building purposes it is very important to know the highest-energy scale at which we can start with TBM mixing and still be consistent with mixing-angle data after running the angles down to the low-energy scale M_Z (the Z-boson mass scale).

The Standard Model (SM) suffers from the hierarchy problem associated with the need to keep electroweak breaking much below the Planck scale. This problem is evaded if the theory is supersymmetric, with supersymmetry broken close to the electroweak scale. For this reason we consider the radiative corrections to neutrino masses and mixing in the context of the minimal supersymmetric extension of the Standard Model (MSSM). We specify the low-energy boundary conditions of the renormalization group equations to be consistent with the three gauge coupling constants and the quark and lepton masses [16]. We assume an effective SUSY scale of $M_S = 500$ GeV. We use the SM renormalization group equations below M_S and the MSSM renormalization group equations above M_S . The only boundary condition set at the family-symmetry breaking scale M_F is exact TBM mixing for the leptons¹. The neutrino masses are set at the low-energy boundary relative to the lightest neutrino mass state (m_{ν_1} with a normal hierarchy and m_{ν_3} with an inverted hierarchy). We keep $|\Delta m_{12}^2|$ the solar mass difference and $|\Delta m_{23}^2|$ the atmospheric mass difference.

Figure 1 shows two contour plots. For the normal hierarchy the plot shows m_{ν_1} versus $\tan\beta$ and for the inverted hierarchy shows m_{ν_3} versus $\tan\beta$. The contours specify $\text{Log}_{10}M_F$ where M_F is the highest-energy family-symmetry breaking scale at which we can set TBM mixing and have the low-energy mixing angles consistent to within 4σ of the low-energy observations. The solar mixing angle θ_{12} is the most sensitive to radiative corrections. Exact TBM mixing gives $\tan^2\theta_{12} = 0.5$, and our 4σ requirement at low energy translates to $\tan^2\theta_{12} = 0.47 \pm 0.2$ [2].

The difference between the two graphs can mostly be understood by the slight bias of the observational data ($\tan^2\theta_{12} < 0.5$) and the opposite directions in which the $\tan^2\theta_{12}$ runs between the normal and inverted hierarchies. Starting with perfect TBM mixing at the scale M_F a normal hierarchy has $\tan^2\theta_{12}$ become

¹We ignore the small departures from TBM at the high scale which may arise from diagonalising the charged-lepton mass matrix [11, 17].

larger as the renormalization scale becomes smaller. Once one falls below M_S , then $\tan^2 \theta_{12}$ begins to get smaller as the renormalization scale goes down during the final leg. The inverted hierarchy has the opposite behavior. Because there is a longer region of supersymmetric running, there is more parameter space of m_{ν_3} versus $\tan \beta$ compatible with $M_F \geq 10^{16}$ GeV. The slight bulge visible in the upper-right of the inverted-hierarchy contour plot with $M_F \approx 10^4$ GeV is due to the opposite directions of the supersymmetric running and the standard model running on $\tan^2 \theta_{12}$ in the region where $M_F \approx M_S$.

The contours in Figure 1 hold implications for QD TBM family-symmetry models. For $m_{\nu_1} > 0.1$ eV, the neutrino spectrum is referred to as quasi-degenerate (QD) [18]². If cosmological observations are considered, they constrain the sum of the neutrinos $\sum_i m_{\nu_i} \leq 0.42$ eV at the 95% confidence level [19]. This implies $m_{\nu_1} \leq 0.14$ eV which excludes the right half of Figure 1. The remaining allowed narrow strip is consistent with the non-observation of neutrinoless double beta decay $\beta\beta_{0\nu}$ which places a limit of $m_{ee} < 0.34$ eV. Uncertainties in nuclear matrix element weaken this bound by about a factor of 3. If we believe that the family-symmetry scale is greater than $M_F > 10^{10}$ GeV, and hypothesize a model which leads to a QD neutrino spectrum with normal hierarchy, then $\tan \beta < 6$ (or $\tan \beta < 8$ for a model with inverted hierarchy). In contrast, if a normal hierarchy model has $\tan \beta > 6$ (or $\tan \beta > 8$ for an inverted hierarchy model), then the lightest neutrino need be less than 0.1 eV and therefore hierarchical.

3 A discrete non-Abelian family symmetry model of QD neutrinos with TBM mixing

As stressed in [20] an underlying $SO(3)$ family symmetry readily leads to a near degenerate neutrino mass spectrum. In their model the chiral superfields, L^i (where i is the $SO(3)$ family index), contain the lepton doublets and transform as triplets under the $SO(3)$ group. The chiral superfields containing the conjugates of the right-handed electron, muon and tau, respectively e^c , μ^c and τ^c , are $SO(3)$ singlets. The effective Majorana neutrino mass is constrained by the symmetry and comes from the superpotential

$$W_{eff} = y_0 (L^i L^i) H_u H_u / M \quad (1)$$

where H_u is the supermultiplet containing the Higgs field whose vacuum expectation value (VEV), $\langle H_u \rangle = v$, is responsible for up quark masses in the MSSM and M is the messenger scale associated with the mechanism generating this dimension 5 term (in the Type II see-saw it is the mass of the exchanged isotriplet Higgs field).

The important point to be taken from eq.(1) is that the family symmetry forces the three light neutrinos to be degenerate. Small departures from degeneracy result when the $SO(3)$ family symmetry is broken. In what follows we will show how this can naturally lead to a mass mixing matrix which gives near TBM mixing. This is done through the breaking of the family symmetry by the non-vanishing vacuum expectation values (VEVs) of familon fields, denoted as ϕ_A^i , where the $A = 3, 23, 123$ labels three distinct fields and serves as a reminder of their VEV directions which are given by

$$\langle \phi_3 \rangle = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \quad \langle \phi_{23} \rangle = \begin{pmatrix} 0 \\ -b \\ b \end{pmatrix} \quad \langle \phi_{123} \rangle = \begin{pmatrix} c \\ c \\ c \end{pmatrix} \quad (2)$$

where a, b and c are complex parameters. Table 1 lists the full set of supermultiplets and their symmetry properties under the $SO(3)$ symmetry extended by a further set of symmetries $G = Z_{3R} \times Z_2 \times U_\tau(1)$ which limit the terms that can appear in the superpotential. Z_{3R} is a discrete R -symmetry which ensures the familon fields are moduli and cannot appear in the superpotential except coupled to “matter” fields carrying non-zero R -charge. The $U_\tau(1)$ symmetry is introduced to distinguish the third family of leptons from the first two. In practice it also explains why the mixing in the charged-lepton sector is different from that in the neutrino sector which leads to near tri-bi-maximal mixing.

The special structure of the VEVs in eq(2) is what will generate TBM mixing and is clearly the most important aspect of the model. This can happen naturally if the underlying family symmetry is not $SO(3)$ but a discrete non-Abelian subgroup. We will discuss below the nature of this symmetry and the vacuum

²We define QD as $m_{\nu_1} > 0.1$ eV because above this value the $\beta\beta_{0\nu}$ constraints for differing hierarchies and phases converge to a common region, as shown in figure 2.

Field	$SO(3)$	Z_{3R}	$U_\tau(1)$	Z_2
L^i	3	1	0	+
e^c	1	1	0	+
μ^c	1	1	0	-
τ^c	1	1	-1	+
$H_{u,d}$	1	0	0	+
ϕ_3^i	3	0	1	+
ϕ_{23}^i	3	0	0	-
ϕ_{123}^i	3	0	0	+
X	1	2	0	-

Table 1: Assignment of the fields under the $SO(3)$ family symmetry.

alignment leading to eq(2) (the X field of Table 1 is introduced to facilitate this vacuum alignment), but first we show that it does generate approximate TBM mixing.

The leading terms in the superpotential responsible for neutrino masses that are invariant under the family symmetries are given by

$$W_\nu = y_0(L^i L^i)H_u H_u + y_\odot(\phi_{123}^i L^i)^2 H_u H_u + y_\oplus(\phi_{23}^i L^i)^2 H_u H_u. \quad (3)$$

where we have suppressed the messenger scale. Note that due to the Z_2 factor there are no cross terms involving $\phi_{23}\phi_{123}$ [21, 22] and due to the $U_\tau(1)$ factor there is no term involving ϕ_3 . As in eq(1), the QD mass scale is set by the first term of eq(3). For near degeneracy, the other terms must be relatively small ($y_\odot c^2, y_\oplus b^2 \ll y_0$, still suppressing the messenger scale).

The charged-lepton masses come from the superpotential

$$W_e = \lambda_e(L^i \phi_{123}^i) e^c H_d + \lambda_\mu(L^i \phi_{23}^i) \mu^c H_d + \lambda_\tau(L^i \phi_3^i) \tau^c H_d. \quad (4)$$

The m_μ/m_τ ratio is given by $\lambda_\mu\langle\phi_{23}^i\rangle/\lambda_d\langle\phi_3^i\rangle$. Using this the mixing between the second and third families of charged leptons is small of $O(m_\mu/m_\tau)$. Similarly one may see that the mixing between the first and second families is of $O(m_e/m_\mu)$ and that between the first and third families is of $O(m_e/m_\tau)$, both very small. Ignoring the small corrections from the charged-lepton sector, the light neutrino mass eigenstates are proportional to the combinations $\phi_{123}^i L^i H_u$ and $\phi_{23}^i L^i H_u$ ³. From eq(2) we see that these are given by

$$\begin{aligned} \nu_\oplus &= \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau) \\ \nu_\odot &= \frac{1}{\sqrt{3}}(\nu_e + \nu_\mu + \nu_\tau) \end{aligned} \quad (5)$$

where $\nu_{e,\mu,\tau}$ are the components of $L^{e,\mu,\tau}$ respectively (selected by the VEV of H_u). Ignoring the small charged-lepton mixings discussed above, $\nu_{e,\mu,\tau}$ can be identified with the current eigenstates. If b and c are real and positive, and $m_\odot = y_\odot c^2 v^2 < m_\oplus = y_\oplus b^2 v^2$, one can see from eq(3) and eq(5) that we obtain the normal hierarchy, in which ν_\oplus may be identified with the atmospheric neutrino with bi-maximal mixing while ν_\odot may be identified with the solar neutrino with tri-maximal mixing. The normal hierarchy persists for a range of complex b and c values in the neighbourhood of the real solution. An inverted hierarchy is possible and viable if b, c are approximately imaginary and real, respectively.

Although here we are working at the effective Lagrangian level, we already noted that $(L^i L^i)HH$ naturally arises from the $SO(3)$ invariant Type II see-saw mechanism. The other two neutrino mass terms can arise from Type I see-saw through exchange of appropriate heavy right-handed Majorana neutrinos, in a manner similarly to that discussed for a $SU(3)$ based model in [24]. Being of different origin it can readily happen that the common mass, $m_0 = y_0 v^2$ is much larger than m_\oplus and m_\odot .

³In finding the mass eigenstates with a complex Majorana mass matrix, one needs to be careful to diagonalize $M_\nu M_\nu^\dagger$ and not just M_ν . Because M_ν is symmetric, it can also be diagonalized by an orthogonal transformation $OM_\nu O^T$. In general $O \neq U_\nu$ and the square of the eigenvalues of M_ν are not the same as those of $M_\nu M_\nu^\dagger$ [23].

	Z_3	Z_2
ϕ^1	ϕ^2	ϕ^1
ϕ^2	ϕ^3	$-\phi^2$
ϕ^3	ϕ^1	$-\phi^3$

Table 2: Action of the group factors Z_3 and Z_2 on the triplet representation ϕ^i .

4 Discrete non-Abelian symmetry and vacuum alignment

We turn now to a discussion as to how the pattern of VEVs displayed in eq(2) is dynamically generated. This can be achieved relatively simply if the underlying family symmetry is a discrete non-Abelian subgroup of $SO(3)$ (and $SU(3)$). A very simple example is given by $A_4 \equiv \Delta(12)$, belonging to the $\Delta(3n^2)$ family of groups [25]. The $\Delta(12)$ invariant terms in the potential are those invariant under the group elements of the semi-direct product $Z_3 \ltimes Z_2$ (which generate the group $\Delta(12)$). The action of these group elements on a triplet representation $\phi^{i=1,2,3}$ is shown in Table 2.

Since $\Delta(12)$ is a subgroup of $SO(3)$, all $SO(3)$ invariants are allowed by the discrete subgroup. Thus the terms of eq(3) and eq(4) are allowed. The discrete subgroup allows additional terms, but these are all higher dimensional and consequently small provided the VEVs of eq(2) are small relative to the relevant messenger mass. Thus the lepton mass and mixing structure discussed is a consequence of the non-Abelian discrete group even though the $SO(3)$ structure used above to motivate it is only approximate.

Turning now to the question of vacuum alignment, consider the leading terms in the potential for the triplet familon fields. Because of the R -symmetry, in the absence of the X -field, there are no F -terms involving just the familon fields coming from the superpotential. The leading D -terms consistent with symmetries of Table 1 are

$$V(\phi) = \alpha m^2 \sum_i |\phi^i|^2 + \beta m^2 \left| \sum_i |\phi^i|^2 \right|^2 + \gamma m^2 \sum_i |\phi^i|^4 + \delta m^2 \left| \sum_i (\phi^i)^2 \right|^2 \quad (6)$$

Here the quadratic term is driven by supersymmetry breaking and m is the gravitino mass. The coefficient includes radiative corrections which can drive it negative at some scale Λ , triggering a VEV for ϕ . The remaining terms can arise through radiative corrections and also only arise if supersymmetry is broken - hence the factor of m^2 on every term. The second term is generated at one-loop order if the superpotential contains a term of the form $\xi Y \sum_i \phi^i \chi^i$ where χ^i and Y are ($Z_{3R} = 1$) massive chiral superfields which we take for presentational simplicity to have mass M . These two terms are invariant under the larger group $SU(3)$ and, if α is negative, generate a VEV of the form $\langle \phi \rangle = (r, s, t)$ where $r^2 + s^2 + t^2 = x^2$, with x^2 a constant of $O(\Lambda^2)$. The third term, consistent with the non-Abelian family group, breaks $SU(3)$ and $SO(3)$. It will be generated if the underlying theory contains a superpotential term of the form $((\phi^1)^2 + \omega^2(\phi^2)^2 + \omega(\phi^3)^2)Z$, where ω is the cube root of unity ($\omega^3 = 1$) and Z is in a singlet representation of $\Delta(12)$ (one of the three irreducible singlet representations and distinct from the representation of Y)⁴. This coupling is invariant under the discrete group but not under $SU(3)$ or $SO(3)$. The resulting third term of eq(6) splits the vacuum degeneracy. For negative α , the minimum for γ positive has $|\langle \phi^i \rangle| = x(1, 1, 1)/\sqrt{3}$ while for γ negative $|\langle \phi^i \rangle| = x(0, 0, 1)$. Finally the fourth term also results from a one-loop radiative correction due to the $\xi Y \sum_i \phi^i \chi^i$ interaction. It is $SO(3)$ but not $SU(3)$ invariant and constrains the phases of the familon fields. For δ negative and γ positive the minimum has $\langle \phi^i \rangle = x(1, 1, 1)/\sqrt{3}$ where x can be complex. This provides a mechanism to generate the vacuum alignment of ϕ_3 and ϕ_{123} as each will have a potential of the form in eq(6) - as we are considering more than one familon, we label the coefficients with the familon's subscript to identify which term they correspond to. The structure of eq(2) results if γ_3 is positive and $\gamma_{123}, \delta_{123}$ are negative.

Finally what about ϕ_{23} ? Its VEV of the form in eq(2) readily results once one includes the effect of the X field of Table 1 because the symmetries allow a term in the superpotential proportional to $X(\phi_{23}\phi_{123})$. This leads to a positive semi-definite term in the potential proportional to $|\phi_{23}\phi_{123}|^2$. There remains the need to align ϕ_{23} and ϕ_3 . This is readily done if radiative corrections generate a term $m^2 \left| \phi_{23}^\dagger \phi_3 \right|^2$ with a negative

⁴One may readily check that it is easy to assign charges under the group G to the new fields, χ^i , Y and Z to allow these couplings.

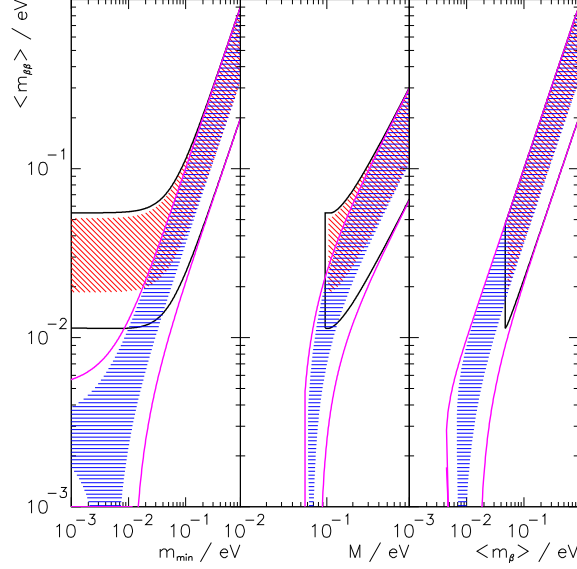


Figure 2: Neutrinoless double-beta decay $m_{\beta\beta}$ plots, from [16]. From left to right, m_{\min} is the absolute value of the lightest neutrino mass, M is the sum of the light neutrino masses, and $\langle m_{\beta} \rangle$ is the average mass determined from low energy beta decays. The shaded areas has width due to the unknown Majorana phases and the areas enclosed by solid lines take into account the errors of oscillation data. The two sets of solid lines correspond to the normal and inverted hierarchies.

coefficient, thus a VEV will develop for ϕ_{23} in the direction given by eq(2)⁵. One may readily check that the higher dimension terms allowed by the symmetry which involve the X field always involve an odd factor of $\phi_{23}\phi_{123}$ and do not disturb the vacuum alignment mechanism discussed above.

5 Neutrinoless double-beta decay

The implication for neutrinoless double-beta decay in this model is unambiguous because the relative phases of the familion fields are determined. The amplitude for neutrinoless double-beta decay is proportional to the magnitude of $\sum m_{\nu_i} U_{ei}^2 \equiv m_{\beta\beta}$ and this is what is measured. For TBM mixing $U_{e\tau}$ vanishes. The relative phase between the remaining two terms is given by $\text{Arg}[m_0 + e^{2ip_{123}} m_{\odot}] - \text{Arg}[m_0]$ where $p_{123} = \text{Arg}[y_{\odot}\phi_{123}\phi_{123}/y_0]$. As $m_{\odot} < m_0$ the relative phase remains small. This corresponds to the upper branches of Figure 2 in the QD region. Complex phases in the VEVs induce other CP violations through the charged-lepton sector that do not significantly affect $m_{\beta\beta}$.

6 Conclusion

Attempts to explain the structure of fermion masses and mixings often rely on structure at a high (Grand Unified?) scale, M_F , to generate the observed pattern. One possibility, consistent with neutrino oscillation, is that neutrinos are nearly degenerate. However, due to enhanced radiative corrections in this case, the observation of near TBM mixing is difficult to reconcile with such a high scale mechanism. To keep the deviations from TBM mixing within experimental limits it is necessary to limit the scale at which TBM mixing is generated. We have determined this scale for the MSSM and found significant bounds on M_F . For example, for degenerate ($m_{\nu_i} > 0.1$ eV), normal hierarchy neutrinos and $\tan\beta > 6$ (or $\tan\beta > 8$, for inverted hierarchy neutrinos), $M_F < 10^{10}$ GeV is required. To get close to the Grand Unified scale with QD neutrinos it is necessary to have very small $\tan\beta$.

⁵Note that in eq(2) we have used the freedom to define the directions such that $\langle \phi_3^{1,2} \rangle = 0$, $\langle \phi_{23}^1 \rangle = 0$.

Turning to the origin of the structure, we have constructed a model based on a discrete non-Abelian family symmetry which gives a QD neutrino spectrum and near TBM mixing. This relies on a natural mechanism for vacuum alignment of the familons which break the family symmetry. The mechanism predicts that neutrinoless double-beta decay should be maximal. Although we only constructed the low-energy effective theory, it fits very well with a see-saw mechanism in which the degenerate mass comes from a Type II see-saw while the small departures from degeneracy are driven by a Type I see-saw.

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